CERN-TH/2000-381

Space-time symmetries and simple superalgebras

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ABSTRACT

We describe spinors in Minkowskian spaces with arbitrary signature and their role in the classification of space-time superalgebras and their R-symmetries in any dimension.

Contribution to the Proceedings of the First Workshop of the RTN Network
"The quantum structure of spacetime and the geometric nature of fundamental interactions"
Berlin, Germany, 4-10 October 2000

CERN-TH/2000-381
December 2000

1 Introduction

We consider supersymmetry algebras in space-times with arbitrary signature and minimal number of spinor generators. The interrelation between super Poincaré and superconformal algebras is elucidated[1]. Minimal superconformal algebras are seen to have as bosonic part a classical semisimple algebra naturally associated to the spin group. This algebra, the Spin(s,t)-algebra[1], depends both on the dimension and on the signature of space time. We also consider superconformal algebras, which are classified by the orthosymplectic algebras.

We then generalize the classification to N-extended space-time superalgebras and notice that R-symmetries may become non-compact depending on the space-time signature[2]. The latter applies to the case of Euclidean super Yang-Mills theories in four dimensions.

2 Properties of spinors of SO(V)

Let V be a real vector space of dimension $D=s+t$ and $\{v_\mu\}$ a basis of it. On V there is a non degenerate symmetric bilinear form which in the basis is given by the matrix

$$\eta_{\mu\nu} = \mathrm{diag}(+,\dots(s \text{ times})\dots,+,-,\dots(t \text{ times})\dots,-).$$

We consider the group Spin(V), the unique double covering of the connected component of SO(s,t) and its spinor representations. A spinor representation of Spin$(V)^{\mathbb{C}}$ is an irreducible complex representation whose highest weights are the fundamental weights corresponding to the right extreme nodes in the Dynkin diagram. These do not descend to representations of SO(V). A spinor type representation is any irreducible representation that doesn't descend to SO(V). A spinor representation of Spin(V) over the reals is an irreducible representation over the reals whose complexification is a direct sum of spin representations[3, 4, 5, 6].

Two parameters, the signature ρ mod(8) and the dimension D mod(8) classify the properties of the spinor representation. Through this paper we will use the following notation,

$$\rho = s-t = \rho_0 + 8n, \qquad D = s+t = D_0 + 8p,$$

where $\rho_0, D_0 = 0,\dots 7$. We set $m = p-n$, so

$$\begin{aligned} s &= \frac{1}{2}(D+\rho) = \frac{1}{2}(\rho_0 + D_0) + 8n + 4m, \\ t &= \frac{1}{2}(D-\rho) = \frac{1}{2}(D_0 - \rho_0) + 4m. \end{aligned}$$

The signature ρ mod(8) determines if the spinor representations are real ($\mathbb{R}$), quaternionic ($\mathbb{H}$) or complex ($\mathbb{C}$) type. Also note that reality properties depend only on $|\rho|$ since Spin(s,t) = Spin(t,s).

The dimension D mod(8) determines the nature of the Spin(V)-morphisms of the spinor representation S. Let $g \in$ Spin(V) and let $\Sigma(g): S \longrightarrow S$ and $L(g): V \longrightarrow V$ the spinor and vector representations of $l \in$ Spin(V) respectively. Then a map A

$$A: S \otimes S \longrightarrow \Lambda^k,$$

where $\Lambda^k = \Lambda^k(V)$ are the k-forms on V, is a Spin(V)-morphism if

$$A(\Sigma(g)s_1 \otimes \Sigma(g)s_2) = L^k(g)A(s_1 \otimes s_2).$$

In Tables 1 and 2, reality and symmetry properties of spinors are reported.

[1]The content of this report is based on Refs. [1] and [2]

ρ_0(odd)	real dim(S)	reality	ρ_0(even)	real dim($S^\pm$)	reality
1	$2^{(D-1)/2}$	$\mathbb{R}$	0	$2^{D/2-1}$	$\mathbb{R}$
3	$2^{(D+1)/2}$	$\mathbb{H}$	2	$2^{D/2}$	$\mathbb{C}$
5	$2^{(D+1)/2}$	$\mathbb{H}$	4	$2^{D/2}$	$\mathbb{H}$
7	$2^{(D-1)/2}$	$\mathbb{R}$	6	$2^{D/2}$	$\mathbb{C}$

Table 1: Reality properties of spinors

D	k even		k odd	
	morphism	symmetry	morphism	symmetry
0	$S^\pm \otimes S^\pm \to \Lambda^k$	$(-1)^{k(k-1)/2}$	$S^\pm \otimes S^\mp \to \Lambda^k$	
1	$S \otimes S \to \Lambda^k$	$(-1)^{k(k-1)/2}$	$S \otimes S \to \Lambda^k$	$(-1)^{k(k-1)/2}$
2	$S^\pm \otimes S^\mp \to \Lambda^k$		$S^\pm \otimes S^\pm \to \Lambda^k$	$(-1)^{k(k-1)/2}$
3	$S \otimes S \to \Lambda^k$	$-(-1)^{k(k-1)/2}$	$S \otimes S \to \Lambda^k$	$(-1)^{k(k-1)/2}$
4	$S^\pm \otimes S^\pm \to \Lambda^k$	$-(-1)^{k(k-1)/2}$	$S^\pm \otimes S^\mp \to \Lambda^k$	
5	$S \otimes S \to \Lambda^k$	$-(-1)^{k(k-1)/2}$	$S \otimes S \to \Lambda^k$	$-(-1)^{k(k-1)/2}$
6	$S^\pm \otimes S^\mp \to \Lambda^k$		$S^\pm \otimes S^\pm \to \Lambda^k$	$-(-1)^{k(k-1)/2}$
7	$S \otimes S \to \Lambda^k$	$(-1)^{k(k-1)/2}$	$S \otimes S \to \Lambda^k$	$-(-1)^{k(k-1)/2}$

Table 2: Properties of morphisms.

3 Orthogonal, symplectic and linear spinors

We consider now morphisms

$$S \otimes S \longrightarrow \Lambda^0 \simeq \mathbb{C}.$$

If a morphism of this kind exists, it is unique up to a multiplicative factor. The vector space of the spinor representation has then a bilinear form invariant under $\mathrm{Spin}(V)$. Looking at Table 2, one can see that this morphism exists except for $D_0 = 2, 6$, where instead a morphism

$$S^\pm \otimes S^\mp \longrightarrow \mathbb{C}$$

occurs.

We shall call a spinor representation orthogonal if it has a symmetric, invariant bilinear form. This happens for $D_0 = 0, 1, 7$ and $\mathrm{Spin}(V)^\mathbb{C}$ (complexification of $\mathrm{Spin}(V)$) is then a subgroup of the complex orthogonal group $\mathrm{SO}(n, \mathbb{C})$, where n is the dimension of the spinor representation (Weyl spinors for D even). The generators of $\mathrm{SO}(n, \mathbb{C})$ are $n \times n$ antisymmetric matrices. These are obtained in terms of the morphisms

$$S \otimes S \longrightarrow \Lambda^k,$$

which are antisymmetric. This gives the decomposition of the adjoint representation of $\mathrm{SO}(n, \mathbb{C})$ under the subgroup $\mathrm{Spin}(V)^\mathbb{C}$. In particular, for $k = 2$ one obtains the generators of $\mathrm{Spin}(V)^\mathbb{C}$.

A spinor representation is called symplectic if it has an antisymmetric, invariant bilinear form. This is the case for $D_0 = 3, 4, 5$. $\mathrm{Spin}(V)^\mathbb{C}$ is a subgroup of the symplectic group $\mathrm{Sp}(2p, \mathbb{C})$, where $2p$ is the dimension of the spinor representation. The Lie algebra $\mathrm{sp}(2p, \mathbb{C})$ is formed by all the symmetric matrices, so it is given in terms of the morphisms $S \otimes S \to \Lambda^k$ which are symmetric. The generators of $\mathrm{Spin}(V)^\mathbb{C}$ correspond to $k = 2$ and are symmetric matrices.

For $D_0 = 2, 6$ one has an invariant morphism

$$B : S^+ \otimes S^- \longrightarrow \mathbb{C}.$$

The representations S^+ and S^- are one the contragradient (or dual) of the other. The spin representations extend to representations of the linear group $\mathrm{GL}(n, \mathbb{C})$, which leaves the pairing B invariant. These spinors are called linear. $\mathrm{Spin}(V)^\mathbb{C}$ is a subgroup of the simple factor $\mathrm{SL}(n, \mathbb{C})$.

These properties depend exclusively on the dimension[6]. When combined with the reality properties, which depend on ρ, one obtains real groups embedded in $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{Sp}(2p,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$ which have an action on the space of the real spinor representation S^σ. The real groups contain as a subgroup $\mathrm{Spin}(V)$.

We need first some general facts about real forms of simple Lie algebras[6]. Let S be a complex vector space of dimension n which carries an irreducible representation of a complex Lie algebra $\mathcal{G}$. Let G be the complex Lie group associated to $\mathcal{G}$. Let σ be a conjugation or a pseudoconjugation on S such that $\sigma X\sigma^{-1}\in\mathcal{G}$ for all $X\in\mathcal{G}$. Then the map

$$X\mapsto X^\sigma=\sigma X\sigma^{-1}$$

is a conjugation of $\mathcal{G}$. We shall write

$$\mathcal{G}^\sigma=\{X\in\mathcal{G}|\mathcal{X}^\sigma=\mathcal{X}\}.$$

$\mathcal{G}^\sigma$ is a real form of $\mathcal{G}$. If $\tau=h\sigma h^{-1}$, with $h\in\mathcal{G}$, $\mathcal{G}^\tau=h\mathcal{G}^\sigma h^{-1}$. $\mathcal{G}^\sigma=\mathcal{G}^{\sigma'}$ if and only if $\sigma'=\epsilon\sigma$ for ϵ a scalar with $|\epsilon|=1$; in particular, if $\mathcal{G}^\sigma$ and $\mathcal{G}^\tau$ are conjugate by G, σ and τ are both conjugations or both pseudoconjugations. The conjugation can also be defined on the group G, $g\mapsto\sigma g\sigma^{-1}$.

4 Real forms of the classical Lie algebras

We describe the real forms of the classical Lie algebras from this point of view[1]. (See also Ref. [7]).

Linear algebra, sl(S).

(a) If σ is a conjugation of S, then there is an isomorphism $S\to\mathbb{C}^n$ such that σ goes over to the standard conjugation of $\mathbb{C}^n$. Then $\mathcal{G}^\sigma\simeq\mathrm{sl}(n,\mathbb{R})$. (The conjugation acting on $\mathrm{gl}(n,\mathbb{C})$ gives the real form $\mathrm{gl}(n,\mathbb{R})$).

(b) If σ is a pseudoconjugation and $\mathcal{G}$ doesn't leave invariant a non degenerate bilinear form, then there is an isomorphism of S with $\mathbb{C}^n$, $n=2p$ such that σ goes over to

$$(z_1,\dots,z_p,z_{p+1},\dots z_{2p})\mapsto(z^*_{p+1},\dots z^*_{2p},-z^*_1,\dots,-z^*_p).$$

Then $\mathcal{G}^\sigma\simeq\mathrm{su}^*(2p)$. (The pseudoconjugation acting in on $\mathrm{gl}(2p,\mathbb{C})$ gives the real form $\mathrm{su}^*(2p)\oplus\mathrm{so}(1,1)$.)

To see this, it is enough to prove that $\mathcal{G}^\sigma$ does not leave invariant any non degenerate hermitian form, so it cannot be of the type $\mathrm{su}(p,q)$. Suppose that $\langle\cdot,\cdot\rangle$ is a $\mathcal{G}^\sigma$-invariant, non degenerate hermitian form. Define $(s_1,s_2):=\langle\sigma(s_1),s_2\rangle$. Then $(\cdot,\cdot)$ is bilinear and $\mathcal{G}^\sigma$-invariant, so it is also $\mathcal{G}$-invariant.

(c) The remaining cases, following E. Cartan's classification of real forms of simple Lie algebras, are $\mathrm{su}(p,q)$, where a non degenerate hermitian bilinear form is left invariant. They do not correspond to a conjugation or pseudoconjugation on S, the space of the fundamental representation. (The real form of $\mathrm{gl}(n,\mathbb{C})$ is in this case $\mathrm{u}(p,q)$).

Orthogonal algebra, so(S). $\mathcal{G}$ leaves invariant a non degenerate, symmetric bilinear form. We will denote it by $(\cdot,\cdot)$.

(a) If σ is a conjugation preserving $\mathcal{G}$, one can prove that there is an isomorphism of S with $\mathbb{C}^n$ such that $(\cdot,\cdot)$ goes to the standard form and $\mathcal{G}^\sigma$ to $\mathrm{so}(p,q)$, $p+q=n$. Moreover, all $\mathrm{so}(p,q)$ are obtained in this form.

(b) If σ is a pseudoconjugation preserving $\mathcal{G}$, $\mathcal{G}^\sigma$ cannot be any of the $\mathrm{so}(p,q)$. By E. Cartan's classification, the only other possibility is that $\mathcal{G}^\sigma\simeq\mathrm{so}^*(2p)$. There is an isomorphism of S with $\mathbb{C}^{2p}$ such that σ goes to

$$(z_1,\dots z_p,z_{p+1},\dots z_{2p})\mapsto(z^*_{p+1},\dots z^*_{2p},-z^*_1,\dots-z^*_p).$$

Symplectic algebra, sp(S). We denote by $(\cdot,\cdot)$ the symplectic form on S.

(a) If σ is a conjugation preserving $\mathcal{G}$, it is clear that there is an isomorphism of S with $\mathbb{C}^{2p}$, such that $\mathcal{G}^\sigma \simeq \mathrm{sp}(2p, \mathbb{R})$.

(b) If σ is a pseudoconjugation preserving $\mathcal{G}$, then $\mathcal{G}^\sigma \simeq \mathrm{usp}(p,q)$, $p+q=n=2m,\ p=2p',\ q=2q'$. All the real forms $\mathrm{usp}(p,q)$ arise in this way. There is an isomorphism of S with $\mathbb{C}^{2p}$ such that σ goes to

$$(z_1, \dots z_m, z_{m+1}, \dots z_n) \mapsto J_m K_{p',q'}(z_1^*, \dots z_m^*, z_{m+1}^*, \dots z_n^*),$$

where

$$J_m = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}, \qquad K_{p',q'} = \begin{pmatrix} -I_{p'\times p'} & 0 & 0 & 0 \\ 0 & I_{q'\times q'} & 0 & 0 \\ 0 & 0 & -I_{p'\times p'} & 0 \\ 0 & 0 & 0 & I_{q'\times q'} \end{pmatrix}.$$

In Section 2 we saw that there is a conjugation on S when the spinors are real and a pseudoconjugation when they are quaternionic[1] (both denoted by σ). We have a group, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{Sp}(2p,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{C})$ acting on S and containing $\mathrm{Spin}(V)^{\mathbb{C}}$. We note that this group is minimal in the classical group series. If the Lie algebra $\mathcal{G}$ of this group is stable under the conjugation

$$X \mapsto \sigma X \sigma^{-1}$$

then the real Lie algebra $\mathcal{G}^\sigma$ acts on S^σ and contains the Lie algebra of $\mathrm{Spin}(V)$. We shall call it the $\mathrm{Spin}(V)$-algebra.

Let B be the space of $\mathrm{Spin}(V)^{\mathbb{C}}$-invariant bilinear forms on S. Since the representation on S is irreducible, this space is at most one dimensional. Let it be one dimensional and let σ be a conjugation or a pseudoconjugation and let $\psi \in B$. We define a conjugation in the space B as

$$\begin{array}{ccc} B & \longrightarrow & B \\ \psi & \mapsto & \psi^\sigma \end{array}$$

$$\psi^\sigma(v,u) = \psi(\sigma(v), \sigma(u))^*.$$

It is then immediate that we can choose $\psi \in B$ such that $\psi^\sigma = \psi$. Then if X belongs to the Lie algebra preserving ψ, so does $\sigma X \sigma^{-1}$.

One can determine the real Lie algebras in each case[1]. All the possible cases must be studied separately. All dimension and signature relations are mod(8). In the following, a relation like $\mathrm{Spin}(V) \subseteq G$ for a group G will mean that the image of $\mathrm{Spin}(V)$ under the spinor representation is in the connected component of G. The same applies for the relation $\mathrm{Spin}(V) \simeq G$. For $\rho =$ 0,1,7 spin algebras commute with a conjugation, for $\rho =$ 3,4,5 they commute with a pseudoconjugation. For $\rho =$ 2,6 they are complex. The complete classification is reported in Table 3.

5 Spin(V) superalgebras

We now consider the embedding of $\mathrm{Spin}(V)$ in simple real superalgebras. We require in general that the odd generators are in a real spinor representation of $\mathrm{Spin}(V)$. In the cases $D_0 = 2, 6$, $\rho_0 = 0, 4$ we have to allow the two independent irreducible representations, S^+ and S^- in the superalgebra, since the relevant morphism is

$$S^+ \otimes S^- \longrightarrow \Lambda^2.$$

The algebra is then non chiral.

We first consider minimal superalgebras[8, 9] i.e. those with the minimal even subalgebra. From the classification of simple superalgebras [10, 11, 12] one obtains the results listed in Table 4.

We note that the even part of the minimal superalgebra contains the $\mathrm{Spin}(V)$ algebra obtained in Section 4 as a simple factor. For all quaternionic cases, $\rho_0 = 3, 4, 5$, a second simple factor SU(2) is present. For the linear cases there is an additional non simple factor, SO(1,1) or U(1), as discussed in Section 4.

Orthogonal $D_0=1,7$	Real, $\rho_0=1,7$	$\mathrm{so}(2^{\frac{(D-1)}{2}},\mathbb{R})$ if $D=\rho$
		$\mathrm{so}(2^{\frac{(D-1)}{2}-1},2^{\frac{(D-1)}{2}-1})$ if $D\neq\rho$
	Quaternionic, $\rho_0=3,5$	$\mathrm{so}^*(2^{\frac{(D-1)}{2}})$
Symplectic $D_0=3,5$	Real, $\rho_0=1,7$	$\mathrm{sp}(2^{\frac{(D-1)}{2}},\mathbb{R})$
	Quaternionic, $\rho_0=3,5$	$\mathrm{usp}(2^{\frac{(D-1)}{2}},\mathbb{R})$ if $D=\rho$
		$\mathrm{usp}(2^{\frac{(D-1)}{2}-1},2^{\frac{(D-1)}{2}-1})$ if $D\neq\rho$
Orthogonal $D_0=0$	Real, $\rho_0=0$	$\mathrm{so}(2^{\frac{D}{2}-1},\mathbb{R})$ if $D=\rho$
		$\mathrm{so}(2^{\frac{D}{2}-2},2^{\frac{D}{2}-2})$ if $D\neq\rho$
	Quaternionic, $\rho_0=4$	$\mathrm{so}^*(2^{\frac{D}{2}-1})$
	Complex, $\rho_0=2,6$	$\mathrm{so}(2^{\frac{D}{2}-1},\mathbb{C})_{\mathbb{R}}$
Symplectic $D_0=4$	Real, $\rho_0=0$	$\mathrm{sp}(2^{\frac{D}{2}-1},\mathbb{R})$
	Quaternionic, $\rho_0=4$	$\mathrm{usp}(2^{\frac{D}{2}-1},\mathbb{R})$ if $D=\rho$
		$\mathrm{usp}(2^{\frac{D}{2}-2},2^{\frac{D}{2}-2})$ if $D\neq\rho$
	Complex, $\rho_0=2,6$	$\mathrm{sp}(2^{\frac{D}{2}-1},\mathbb{C})_{\mathbb{R}}$
Linear $D_0=2,6$	Real, $\rho_0=0$	$\mathrm{sl}(2^{\frac{D}{2}-1},\mathbb{R})$
	Quaternionic, $\rho_0=4$	$\mathrm{su}^*(2^{\frac{D}{2}-1})$
	Complex, $\rho_0=2,6$	$\mathrm{su}(2^{\frac{D}{2}-1})$ if $D=\rho$
		$\mathrm{su}(2^{\frac{D}{2}-2},2^{\frac{D}{2}-2})$ if $D\neq\rho$

Table 3: Spin(s,t) algebras.

For $D=7$ and $\rho=3$ there is actually a smaller superalgebra, the exceptional superalgebra $f(4)$ with bosonic part spin(5,2)×su(2). The superalgebra appearing in Table 4 belongs to the classical series and its even part is so*(8)×su(2), being so*(8) the Spin$(5,2)$-algebra.

Keeping the same number of odd generators, the maximal simple superalgebra containing Spin(V) is an orthosymplectic algebra with Spin$(V)\subset$ Sp$(2n,\mathbb{R})$, being $2n$ the real dimension of S. The various cases are displayed in the Table 5. We note that the minimal superalgebra is not a subalgebra of the maximal one, although it is so for the bosonic parts.

6 Extended Superalgebras

The present analysis can be generalized to the case of N copies of the spinor representation of spin (s,t)-algebras[2]. By looking at the classification of classical simple superalgebras[8]–[13], we find extensions for all N, where the number of supersymmetries is always even if spinors are quaternionic (because of reality properties) or orthogonal (because of symmetry properties).

In Table 6 the classification analogous to the one in Table 4 is given. SuperPoincaré algebras can be obtained from the simple superalgebras either by contraction Spin$(s,t)\rightarrow$ InSpin$(s,t-1)$ or as subalgebras Spin$(s,t)\rightarrow$ InSpin$(s-1,t-1)$. It is important to observe that the R-symmetry may be non-compact for different signatures of space-time.

In fact the conjugation properties of the R-symmetry algebra is the same of the space-time part.

As an example if we consider Euclidean four-dimensional $N=2$ and $N=4$ Yang-Mills theory, the R-symmetry becomes respectively SU(2) × SO(1,1) and SU*(4). The first case was considered long ago by Zumino[14]. These are the superalgebras appropriate for Yang-Mills instantons. On the other hand, if we consider a Minkowskian space with signature (2,2) the R-symmetry is $GL(2,\mathbb{R})$ (for $N=2$) and $SL(4,\mathbb{R})$ for $N=4$.

Compact R-symmetries occur for $q=0$ in Table 6, including all cases when the conformal group SO(D,2) corresponds to ordinary Minkowski space $V_{(D-1,1)}$.

D_0	ρ_0	Spin(V) algebra	Spin(V) superalgebra
1,7	1,7	$\mathrm{so}(2^{(D-3)/2}, 2^{(D-3)/2})$	
1,7	3,5	$\mathrm{so}^*(2^{(D-1)/2})$	$\mathrm{osp}(2^{(D-1)/2})^*\lvert 2)$
3,5	1,7	$\mathrm{sp}(2^{(D-1)/2}, \mathbb{R})$	$\mathrm{osp}(1\lvert 2^{(D-1)/2}, \mathbb{R})$
3,5	3,5	$\mathrm{usp}(2^{(D-3)/2}, 2^{(D-3)/2})$	
0	0	$\mathrm{so}(2^{(D-4)/2}, 2^{(D-4)/2})$	
0	2,6	$\mathrm{so}(2^{(D-2)/2}, \mathbb{C})^{\mathbb{R}}$	
0	4	$\mathrm{so}^*(2^{(D-2)/2})$	$\mathrm{osp}(2^{(D-2)/2})^*\lvert 2)$
2,6	0	$\mathrm{sl}(2^{(D-2)/2}, \mathbb{R})$	$\mathrm{sl}(2^{(D-2)/2}\lvert 1)$
2,6	2,6	$\mathrm{su}(2^{(D-4)/2}, 2^{(D-4)/2})$	$\mathrm{su}(2^{(D-4)/2}, 2^{(D-4)/2}\lvert 1)$
2,6	4	$\mathrm{su}^*(2^{(D-2)/2}))$	$\mathrm{su}(2^{(D-2)/2})^*\lvert 2)$
4	0	$\mathrm{sp}(2^{(D-2)/2}, \mathbb{R})$	$\mathrm{osp}(1\lvert 2^{(D-2)/2}, \mathbb{R})$
4	2,6	$\mathrm{sp}(2^{(D-2)/2}, \mathbb{C})^{\mathbb{R}}$	$\mathrm{osp}(1\lvert 2^{(D-2)/2}, \mathbb{C})$
4	4	$\mathrm{usp}(2^{(D-4)/2}, 2^{(D-4)/2})$	

Table 4: Minimal Spin(V) superalgebras.

D_0	ρ_0	Orthosymplectic
3,5,	1,7	$\mathrm{osp}(1\lvert 2^{(D-1)/2}, \mathbb{R})$
1,7	3,5	$\mathrm{osp}(1\lvert 2^{(D+1)/2}, \mathbb{R})$
0	4	$\mathrm{osp}(1\lvert 2^{D/2}, \mathbb{R})$
4	0	$\mathrm{osp}(1\lvert 2^{(D-2)/2}, \mathbb{R})$
4	2,6	$\mathrm{osp}(1\lvert 2^{D/2}, \mathbb{R})$
2,6	0	$\mathrm{osp}(1\lvert 2^{D/2}, \mathbb{R})$
2,6	4	$\mathrm{osp}(1\lvert 2^{(D+2)/2}, \mathbb{R})$
2,6	2,6	$\mathrm{osp}(1\lvert 2^{D/2}, \mathbb{R})$

Table 5: Orthosymplectic Spin(V) superalgebras

D_0	ρ_0	R-symmetry	Spin(s,t) superalgebra
1,7	1,7	$\mathrm{sp}(2N, \mathbb{R})$	$\mathrm{osp}(2^{\frac{D-3}{2}}, 2^{\frac{D-3}{2}}\lvert 2N, \mathbb{R})$
1,7	3,5	$\mathrm{usp}(2N-2q, 2q)$	$\mathrm{osp}(2^{\frac{D-1}{2}}{}^*\lvert 2N-2q, 2q)$
3,5	1,7	$\mathrm{so}(N-q, q)$	$\mathrm{osp}(N-q, q\lvert 2^{\frac{D-1}{2}})$
3,5	3,5	$\mathrm{so}^*(2N)$	$\mathrm{osp}(2N^*\lvert 2^{\frac{D-3}{2}}, 2^{\frac{D-3}{2}})$
0	0	$\mathrm{sp}(2N, \mathbb{R})$	$\mathrm{osp}(2^{\frac{D-4}{2}}, 2^{\frac{D-4}{2}}\lvert 2N)$
0	2,6	$\mathrm{sp}(2N, \mathbb{C})_{\mathbb{R}}$	$\mathrm{osp}(2^{\frac{D-2}{2}}\lvert 2N, \mathbb{C})_{\mathbb{R}}$
0	4	$\mathrm{usp}(2N-2q, 2q)$	$\mathrm{osp}(2^{\frac{D-2}{2}}{}^*\lvert 2N-2q, 2q)$
2,6	0	$\mathrm{sl}(N, \mathbb{R})$	$\mathrm{sl}(2^{\frac{D-2}{2}}\lvert N, \mathbb{R})$
2,6	2,6	$\mathrm{su}(N-q, q)$	$\mathrm{su}(2^{\frac{D-4}{2}}, 2^{\frac{D-4}{2}}\lvert N-q, q)$
2,6	4	$\mathrm{su}^*(2N, \mathbb{R})$	$\mathrm{su}(2^{\frac{D-2}{2*}}\lvert 2N^*)$
4	0	$\mathrm{so}(N-q, q)$	$\mathrm{osp}(N-q, q\lvert 2^{\frac{D-2}{2}})$
4	2,6	$\mathrm{so}(N, \mathbb{C})_{\mathbb{R}}$	$\mathrm{osp}(N\lvert 2^{\frac{D-2}{2}}, \mathbb{C})_{\mathbb{R}}$
4	4	$\mathrm{so}^*(2N)$	$\mathrm{osp}(2N^*\lvert 2^{\frac{D-4}{2}}, 2^{\frac{D-4}{2}})$

Table 6: N-extended Spin(s,t) superalgebras.

Acknowledgements

This lecture is based on collaborations with R. D'Auria and M.A. Llédo and R. Varadarajan. Enlightning conversations with A. van Proeyen and R. Stora are also acknowledged. This work has been supported in part by the European Commission RTN network HPRN-CT-2000-00131, (Laboratori Nazionali di Frascati, INFN) and by the D.O.E. grant DE-FG03-91ER40662, Task C.

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